A Classification of 6R Manipulators

Ming-Zhe Chen
Qwest Communications Inc.
5751 Sells Mill Dr
Dublin, OH 43017, USA

February 1, 2008

Abstract

This paper presents a classification of generic 6-revolute jointed (6R) manipulators using homotopy class of their critical point manifold. A part of classification is listed in this paper because of the complexity of homotopy class of 4-torus. The results of this classification will serve future research of the classification and topological properties of maniplators joint space and workspace.

1 Introduction

In robotics, the topological properties of manipulator joint space and workspace are related to the singularities. A lot of literature deal with the manipulator singularities [2] [17] [8] [13] [10] [16] [14] [12] [3] [11] [15] and references therein. The kinematics of manipulator can be described as a smooth map:

$$\Phi : X \mapsto Y$$

where X denotes the space of joint variables (x_1 x_2 x_3 x_4 x_5 x_6) for 6R manipulator and Y is the workspace of positions and orientations of the end-effector. Manipulator with m revolte joint, the joint space is

$$X = T^m$$

where T^m denotes the m-dimensional torus. The workspace Y can be special euclidean group SE(3):

$$SE(3) \equiv SO(3) \times R^3$$

which is a Lie group, that is a semidirect product of the special orthogonal group SO(3) and R^3. From the map Φ we can work out the Jacobian, which is the linear map k on the tangent spaces:

$$k : R^m \mapsto R^n$$

where m is the dimension of joint space and n is the dimension of workspace, in this paper m = n = 6. The Jacobian J will be related to:

$$\begin{pmatrix} \omega \\ v \end{pmatrix} = J\dot{X}$$

where ω is the angular velocity of the end-effector, v is the velocity of the end-effector and $\dot{X}$ is the joint rate. Let r = rank(J), then the singular set for the map k can be defined as:

$$S_k = \{x \in R^m | \, r < min(m,n)\}$$

The singular set can also be decomposed into strata [7] [12]:

$$S_k = \bigcup_{c=1}^{min(m,n)} S_c$$

where

$$S_c = \{x \in R^m |\ r = min(m, n) - c\}$$

where the number c is called the corank of the map k. A manipulator is said generic if its singularities are generic, that is, if they form smooth manifolds in joint space. Non-generic manipulaters often arise from geometric simplication condition in the structue. Generic manipulators have only ordinary singularities [2] [17] [10], that is, the singular set (manifolds) behaves like a regular surface, i.e., corank c = 1. In this paper, we deal with generic 6R manipulators. So the higher order singularities do not exist for this kind of 6R manipulators, which can be interpreted as bifurcation. In fact, all possible singular configurations are the zero set of det(J):

$$det(J) = 0$$

After some algebraic manipulation, letting $q_1 = tan(x_1/2)$, $q_2 = tan(x_2/2)$, $q_3 = tan(x_3/2)$, $q_4 = tan(x_4/2)$, $q_5 = tan(x_5/2)$, $q_6 = tan(x_6/2)$, above equation can be converted to the form:

$$f_1(q_2, q_3, q_4)q_5^8 + \ldots = 0$$

or

$$f_2(q_2, q_3, q_5)q_4^{10} + \ldots = 0$$

or

$$f_3(q_2, q_4, q_5)q_3^{12} + \ldots = 0$$

or

$$f_4(q_3, q_4, q_5)q_2^{14} + \ldots = 0$$

The result is too complicated to be displayed here. We just show the first term of the equation. In above equations, we use Denavit-Hartenberg parameters. The above equations may have "zero at infinity" (singularities at infinity). We do not pay extra attention to it since these singularities are a part of the generic singular surface. Since above equations are independent of q_1 and q_6, the singular surface is projected onto torus T^4. The singular surface forms branches, i.e., the connected components, on the surface of T^4. In the following section we will classify the manipulator by the singular surface forming branches, which are the group of homotopy class in T^4.

2 A Classification of 6R Manipulators

Singular surface can be characterized by their fundemental group of homotopy class in T^4. The fundemental group of T^4 is the group of loop equivalence classes, denoted by $\pi_1(T^4)$ [9]:

$$\pi_1(T^4) = \pi_1(T^2) \times \pi_1(T^2) = Z \times Z \times Z \times Z$$

where Z is the set of integer. Each element of $\pi_1(T^4)$, which is a set of homotopicly equivalent singular surfaces, can be labeled by integers (I_2,I_3,I_4,I_5), which characterize how many integral times the "curve" "wrap around" the generator of T^4. It is known a class of 6R manipulators which have the same homotopy class will have similar topological properties in the their joint spaces. Singular manifolds can divide the joint space of 6R manipulator in at least two singularity-free domain called c-sheets. A single singular manifold branch, which can cut the joint space into two c-sheets, is said to be separating; otherwise, it is said non-separating, which must combine with other branch to divide the joint space. In the following, we enumerate the branch homotopy classes.

(1) (0,0,0,0) homotopy class: the branch is separating and can appear alone. The result is due to the topology of the torus.

(2) 2(0,0,0,0), 3(0,0,0,0) and 4(0,0,0,0) homotopy classes: the branches are separating.

But more than 4 coexisting (0,0,0,0) branches would yield more than 8 intersections with the generator, here 8 is the minimum of the maximum allowable times around the generator. (1,0,0,0) or (0,1,0,0) or (0,0,1,0) or (0,0,0,1) cannot divide the torus, so they cannot appear alone. (I_2,0,0,0) or (0,I_3,0,0) or (0,0,I_4,0) or (0,0,0,I_5), when $I_2 > 1$ or $I_3 > 1$ or $I_4 > 1$ or $I_5 > 1$, are impossible because these helical curves must go backward, which cannot be done without self intersections. Since $\pi_1(T^4) = \pi_1(T^2) \times \pi_1(T^2)$, we can consider the singular surfaces have two branches each of which is on one torus T^2. Possible two branches are (I_2,I_3,0,0) + (0,0,I_4,I_5), here $I_2 \leq 12$, $I_3 \leq 14$, $I_4 \leq 8$, and $I_5 \leq 10$, because the degrees of q_2, q_3, q_4, and q_5 in the equation det(J) = 0 set up the maximum allowable times for I_2, I_3, I_4, and I_5 around the generator of T^2. But either (I_2,I_3,0,0) or (0,0,I_4,I_5) cannot be separating because they form helical closed bands. In paper [17], there are eight homotopy classes for torus T^2, that is, (0,0), 2(0,0), (0,0)+2(1,0), 2(1,0), 4(1,0), 2(0,1), 2(1,1), and 2(2,1). We denote these classes to be H2. So following classes are homotopy classes which are separating branches:

(3) (H2,0,0) + (0,0,I_4,I_5), (I_2,I_3,0,0) + (0,0,H2) and (H2,0,0) + (0,0,H2) homotopy classes: the branches are separating.

For the torus T^2 mentioned in this paper, there are more separating homotopy classes. For torus (I_2,I_3,0,0), (11,14,0,0) + (1,0,0,0), (10,14,0,0) + 2(1,0,0,0), (9,14,0,0) + 3(1,0,0,0), (8,14,0,0) + 4(1,0,0,0), (7,14,0,0) + 5(1,0,0,0), (6,14,0,0) + 6(1,0,0,0), (5,14,0,0) + 7(1,0,0,0), (4,14,0,0) + 8(1,0,0,0), (3,14,0,0) + 9(1,0,0,0), (2,14,0,0) + 10(1,0,0,0), (1,14,0,0) + 11(1,0,0,0) and (12,13,0,0) + (0,1,0,0), (12,12,0,0) + 2(0,1,0,0), (12,11,0,0) + 3(0,1,0,0), (12,10,0,0) + 4(0,1,0,0), (12,9,0,0) + 5(0,1,0,0), (12,8,0,0) + 6(0,1,0,0), (12,7,0,0) + 7(0,1,0,0), (12,6,0,0) + 8(0,1,0,0), (12,5,0,0) + 9(0,1,0,0), (12,4,0,0) + 10(0,1,0,0), (12,3,0,0) + 11(0,1,0,0), (12,2,0,0) + 12(0,1,0,0), (12,1,0,0) + 13(0,1,0,0). are separating homotopy classes. We denote these classes to be H3. Similarly, for torus (0,0,I_4,I_5), (0,0,7,10) + (0,0,1,0), (0,0,6,10) + 2(0,0,1,0), (0,0,5,10) + 3(0,0,1,0), (0,0,4,10) + 4(0,0,1,0), (0,0,3,10) + 5(0,0,1,0), (0,0,2,10) + 6(0,0,1,0), (0,0,1,10) + 7(0,0,1,0) and (0,0,8,9) + (0,0,0,1), (0,0,8,8) + 2(0,0,0,1), (0,0,8,7) + 3(0,0,0,1), (0,0,8,6) + 4(0,0,0,1), (0,0,8,5) + 5(0,0,0,1), (0,0,8,4) + 6(0,0,0,1), (0,0,8,3) + 7(0,0,0,1), (0,0,8,2) + 8(0,0,0,1), (0,0,8,1) + 9(0,0,0,1). are separating homotopy classes. We denote these classes to be H4. So,

(4) H3 and H4 homotopy classes: the branches are separating.

There are other combinations such as (0,0,6,8) + (0,0,1,0) etc. we cannot enumerate here. But the combination like (0,0,7,9) + (0,0,1,0) + (0,0,0,1) would lead to intersecting branches: (0,0,1,0) and (0,0,0,1), and it cannot exist. In fact, in order to get a class of 6R manipulators which have the same homotopy class, i.e., to get a set of parameters of the manipulators, we must solve the equation det(J) = 0 under some conditions. For example, for (0,0,0,0) homotopy class, we must solve above equation to get a set of parameters under the conditions:

$$-\pi < q_2 < \pi$$

$$-\pi < q_3 < \pi$$

$$-\pi < q_4 < \pi$$

$$-\pi < q_5 < \pi$$

and for 2(0,0,0,0) homotopy class, we must solve equation det(J) = 0 to get two branches solution and a set of parameters under above conditions. There are still a lot of combinations we cannot enumerate here. It indicates the complexity of the joint space of 6R manipulator. Recently, there are some papers [6] [5] [1] [4] dealing with the topological properties of manipulators, but we still have little understanding of the topological properties of joint space and workspace of the manipulator.

References

[1] C. Altafini. Redundant robotic chains on riemannian submersions. *IEEE Transactions on Robotics and Automation*, 20(2):335–40, 2004.

[2] J. W. Burdick. A classification of 3r positioning manipulator singularities and geometries. *Journal of Mechanisms and Machine Theory*, 30:71–89, 1995.

[3] Ming-Zhe Chen. A study of the jacobian matrix of robots (in chinese). *Chinese Society of Automation, Conference of Robotics*, 1988.

[4] B. Aronov et al. Motion planning for multiple robots. *Proc. 14th Symp. Computational Geometry, ACM*, pages 374–82, 1998.

[5] Michael Farber. Topological complexity of motion planning. *Discrete and Computational Geometry*, 29:211–221, 2003.

[6] Michael Farber. Collision free motion planning on graphs. *Algorithmic Foundations Of Robotics*, VI:123–138, 2005.

[7] M. Golubitsky and V. Guillemin. *Stable Mapping and Their Singularities.* Springer, New York, 1973.

[8] K. C. Gupta and B. Roth. Design considerations for manipulator workspaces. *ASME Journal of Mechanical Design*, 104:704–711, 1982.

[9] A. Hatcher. *Algebraic Topology.* Cambridge University Press, 2001.

[10] A. Karger. Singularity analysis of serial robot-manipulators. *ASME Journal of Mechanical Design*, 118:520–25, 1996.

[11] Jon Kieffer. Differential analysis of bifurcations and isolated singularities for robots and mechanisms. *IEEE Transactions on Robotics and Automation*, 10(1):1–10, 1994.

[12] T. Krzysztof. Differential topology of the inverse kinematic problem for redundant robot manipulators. *The Intern. Journ. of Robotic Research*, 10(5):492–504, 1991.

[13] A. Kumar and K. J. Waldron. The workspace of mechanical manipulators. *ASME Journal of Mechanical Design*, 103:665–672, 1981.

[14] F. L. Litvin and J. Tan. Singularities in motion and displacement functions of constrained mechanical systems. *The Intern. Journ. of Robotics Research*, 8(2):30–43, 1989.

[15] J. M. Selig. *Geometric Fundamentals Of Robotics.* Springer, New York, second edition, 2005.

[16] K. Sugimoto and J. Duffy. Determination of extreme distances of a robot-hand, part 1 and 2. *ASME Journal of Mechanical Design*, 103:631–36, 776–83, 1981.

[17] P. Wenger. Classification of 3r positioning manipulators. *ASME Journal of Mechanical Design*, 120:327–332, 1998.